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Triangle-free triangulations

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ABSTRACT

The flip operation on colored inner-triangle-free triangulations of a convex polygon is studied. It is shown that the affine Weyl group $\widetilde{C}_n$ acts transitively on these triangulations by colored flips, and that the resulting colored flip graph is closely related to a lower interval in the weak order on $\widetilde{C}_n$. Lattice properties of this order are then applied to compute the diameter.

1. Introduction

In a seminal paper, using volume computations in hyperbolic geometry, Sleator, Tarjan and Thurston [7] computed the diameter of the flip graph of all triangulations of a convex n-gon, for large n. For special classes of triangulations, the diameter problem – and, sometimes, even the question of connectivity – is still open. For instance, monochromatic-triangle-free triangulations were introduced by Propp. Sagan [6] showed that the corresponding flip graph is connected only if two colors are used. The diameter in this case is not known.

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This paper studies the set of inner-triangle-free triangulations, which is contained in the set of monochromatic-triangle-free triangulations. Methods from Coxeter group theory are applied to describe the structure of the resulting colored flip graph and to compute its diameter. It is shown that the affine Weyl group $\widetilde{C}_n$ acts transitively, by flips, on such triangulations. The stabilizer is computed, leading to an interpretation of the flip graph as a Schreier graph. This graph is closely related to a distinguished lower interval in the weak order on $\widetilde{C}_n$. Lattice properties of this order are then applied to compute the diameter.

2. Basic concepts

Label the vertices of a convex $(n+4)$-gon P_{n+4} $(n>0)$ by the elements $0,\dots,n+3$ of the additive cyclic group $\mathbb{Z}_{n+4}$. Consider a triangulation (with no extra vertices) of the polygon. Each edge of the polygon is called an *external edge* of the triangulation; all other edges of the triangulation are called *internal edges*, or *chords*.

Definition 2.1. A triangulation of a convex $(n+4)$-gon P_{n+4} is called *inner-triangle-free* (or simply *triangle-free*) if it contains no triangle with 3 internal edges. The set of all triangle-free triangulations of P_{n+4} is denoted $\mathit{TFT}(n)$.

Definition 2.2. A chord in P_{n+4} is called *short* if it connects the vertices labeled $i-1$ and $i+1$, for some $i \in \mathbb{Z}_{n+4}$.

Claim 2.3. *For $n>0$, a triangulation of P_{n+4} is triangle-free if and only if it contains only two short chords.*

Proof. Any triangulation of P_{n+4} consists of $n+1$ diagonals and $n+2$ triangles. Each chord lies in exactly 2 triangles. Thus the average number of chords per triangle is $\frac{2(n+1)}{n+2} = 2 - \frac{2}{n+2}$. By definition, a triangulation is triangle-free if and only if each triangle contains at most 2 chords. On the other hand, each triangle contains at least one chord. One concludes that there are exactly two triangles each containing only one chord, completing the proof. □

Remark 2.4. Recall that the planar dual tree of a triangulation is the graph whose nodes are all bounded triangular regions, where two nodes are connected by an arc if the intersection of the corresponding regions is a chord. A triangulation is triangle-free if and only if the dual tree is a path. Short chords in a given triangulation correspond to leaves of the dual tree. Hence, the number of short chords is at least two, and is exactly two if and only if the dual tree is a path, yielding an alternative proof for Claim 2.3. See also [2].

Definition 2.5. A *proper coloring* of a triangulation $T \in \mathit{TFT}(n)$ is a labeling of the chords by $0,\dots,n$ in the following inductive way: Choose a short chord and label it 0. Inductively, a chord which was not yet labeled and is contained in a triangle whose other chord has been labeled i, is labeled $i+1$.

It is easy to see that this uniquely defines the coloring. The set of all properly colored triangle-free triangulations is denoted $\mathit{CTFT}(n)$.

Definition 2.6. Each chord in a triangulation is a diagonal of a unique quadrangle (the union of two adjacent triangles). Replacing this chord by the other diagonal of that quadrangle is a *flip* of the chord.

The *colored flip graph* Γ_n is defined as follows: the nodes are all the colored triangle-free triangulations in $\mathit{CTFT}(n)$. Two triangulations are connected in Γ_n by an arc labeled i if one is obtained from the other by a flip of the chord labeled i (see Figs. 2.1 and 2.2).

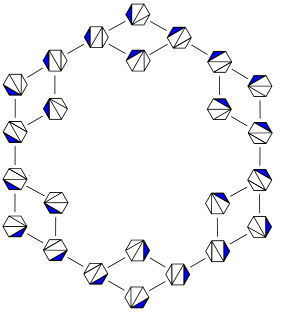

Fig. 2.1. Γ_2 (without edge labels).

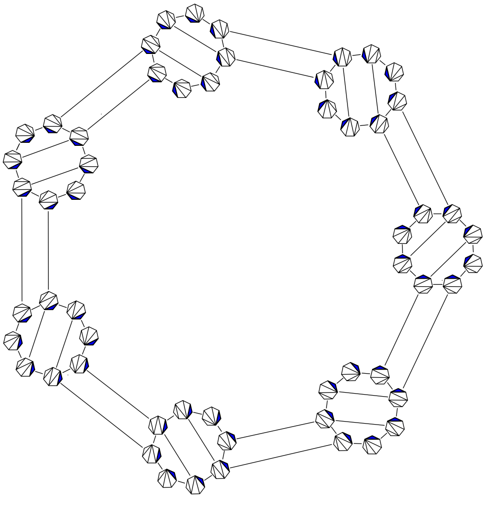

Fig. 2.2. Γ_3 (without edge labels).

By Claim 2.3,

Corollary 2.7. *For $n > 0$, any triangle-free triangulation of P_{n+4} has exactly two proper colorings. In other words,*

$$\#CTFT(n) = 2 \cdot \#TFT(n).$$

Definition 2.8. Define a map

$$\varphi : CTFT(n) \to \mathbb{Z}_{n+4} \times \mathbb{Z}_2^n$$

as follows: Let $T \in CTFT(n)$. If the (short) chord labeled 0 in T is $[a-1, a+1]$ for $a \in \mathbb{Z}_{n+4}$, let $\varphi(T)_0 := a$. For $1 \leqslant i \leqslant n$, assume that the chord labeled $i-1$ in T is $[a-k, a+m]$ for some $k, m \geqslant 1$, $k+m = i+1$. The chord labeled i is then either $[a-k-1, a+m]$ or $[a-k, a+m+1]$. Let $\varphi(T)_i$ be 0 in the former case and 1 in the latter.

Observation 2.9. φ is a bijection.

Corollary 2.10. *For $n > 0$, the number of properly colored triangle-free triangulations of a convex $(n+4)$-gon is*

$$\#CTFT(n) = (n+4) \cdot 2^n.$$

Remark 2.11. Following Remark 2.4, one may observe that, for $n > 0$ a proper coloring of a triangle-free triangulation is just one of the two possible linear orderings of the edges of the dual path.

3. Group action by flips

In this section we assume that $n > 1$, in order to obtain a uniform description of the group actions.

3.1. The $\widetilde{C}_n$-action

Let $\widetilde{C}_n$ be the affine Weyl group generated by

$$S = \{s_0, s_1, \ldots, s_{n-1}, s_n\}$$

subject to the Coxeter relations

$$s_i^2 = 1 \quad (\forall i), \tag{1}$$

$$(s_i s_j)^2 = 1 \quad \big(|j-i| > 1\big), \tag{2}$$

$$(s_i s_{i+1})^3 = 1 \quad (1 \leqslant i < n-1), \tag{3}$$

and

$$(s_i s_{i+1})^4 = 1 \quad (i = 0, n-1). \tag{4}$$

The group $\widetilde{C}_n$ acts naturally on $CTFT(n)$ by flips: generator s_i flips the chord labeled i in $T \in CTFT(n)$, provided that the resulting colored triangulation still belongs to $CTFT(n)$. If this is not the case, T is unchanged by s_i.

Notice that $s_i(T) = T$ if and only if $\varphi(T)_i = \varphi(T)_{i+1}$; also the only short chords are labeled by 0 and n, hence s_0 and s_n never leave the corresponding chords unchanged. Furthermore, one can easily verify that by definition, the following observation holds.

Observation 3.1. For every $T \in CTFT(n)$

$$\big(\varphi(s_0 T)\big)_j = \begin{cases} \varphi(T)_j, & \text{if } j \neq 0, 1, \\ \varphi(T)_0 + 1 \bmod n+4, & \text{if } j = 0 \text{ and } \varphi(T)_1 = 0, \\ \varphi(T)_0 - 1 \bmod n+4, & \text{if } j = 0 \text{ and } \varphi(T)_1 = 1, \\ \varphi(T)_1 + 1 \bmod 2, & \text{if } j = 1 \text{ and } \varphi(T)_1 = 0; \end{cases}$$

$$\big(\varphi(s_n T)\big)_j = \begin{cases} \varphi(T)_j, & \text{if } j \neq n, \\ \varphi(T)_n + 1 \bmod 2, & \text{if } j = n; \end{cases}$$

and

$$\big(\varphi(s_i T)\big)_j = \varphi(T)_{s_i(j)} \quad (0 < i < n).$$

Proposition 3.2. *This operation determines a transitive $\widetilde{C}_n$-action CTFT(n).*

Proof. To prove that the operation is a $\widetilde{C}_n$-action, it suffices to show that it is consistent with the defining Coxeter relations of $\widetilde{C}_n$. Indeed, for every i, s_i acts on a particular triangulation $T \in CTFT(n)$ by flipping the diagonal labeled by i or leaving it unchanged; in both cases $s_i^2(T) = T$. If $|j-i| > 1$, s_i and s_j act on diagonals of quadrangles with no common triangle, hence s_i and s_j commute. Thus relation (2) is satisfied. Relations (3) and (4) may be verified by a direct calculation of the corresponding flip operation, taking in account the relative position of the relevant chords. Alternatively, all relations may be easily verified using Observation 3.1. We leave verification of the details to the reader.

To prove that the action is transitive, notice first that s_0 changes the location of the chord labeled by 0, where the cyclic orientation of this change depends on the relative position of the chord labeled by 1. It, thus, suffices to prove that the maximal parabolic subgroup of $\widetilde{C}_n$, $\langle s_1, \dots, s_n\rangle$ acts transitively on all colored triangle-free triangulations with a given 0 chord. Indeed, the parabolic subgroup $\langle s_1, \dots, s_n\rangle$ is isomorphic to the classical Weyl group B_n. By Observation 3.1, the restricted B_n-action on all colored triangle-free triangulations with a given 0 chord, may be identified with the natural B_n-action on all subsets of $\{1, \dots, n\}$, and is thus transitive. $\square$

3.2. Stabilizer

Define

$$g_0 := s_0 s_1 \cdots s_{n-2} s_n s_{n-1} s_n s_{n-2} \cdots s_1 s_0 \in \widetilde{C}_n$$

and

$$g_n := (s_n \cdots s_0)^{n+4} \in \widetilde{C}_n.$$

Denote:

$$T_0 := \varphi^{-1}(0, \underbrace{0, \dots, 0}_{n \text{ times}}),$$

the *canonical colored star triangulation.*

Theorem 3.3. *The subgroup $St_n = \langle g_0, s_1, \dots, s_{n-1}, g_n\rangle$ of $\widetilde{C}_n$ is the stabilizer, under the $\widetilde{C}_n$-action on CTFT(n), of the canonical colored star triangulation T_0. Stabilizers of other colored triangulations are subgroups of $\widetilde{C}_n$ conjugate to St_n.*

Proof. We shall proceed by a volume argument, using a sequence of technical observations.

Consider the action of $\widetilde{C}_n$ on $\mathbb{R}^n$ given by

$$s_0(x_1, x_2, \dots, x_n) := (-x_1, x_2, \dots, x_n),$$

$$s_i(\dots, x_i, x_{i+1}, \dots) := (\dots, x_{i+1}, x_i, \dots) \quad (1 \leqslant i \leqslant n-1),$$

$$s_n(x_1, \dots, x_{n-1}, x_n) := (x_1, \dots, x_{n-1}, 2 - x_n).$$

It is well known that this gives rise to a faithful n-dimensional affine representation of $\widetilde{C}_n$ (the natural action of $\widetilde{C}_n$ on its root space). The reflecting hyperplanes for the reflections s_i are

$$H_0 := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\},$$
$$H_i := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_{i+1}\} \quad (1 \leqslant i \leqslant n-1),$$
$$H_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 1\}.$$

Observation 3.4. A fundamental region for the above action of $\widetilde{C}_n$ is the n-dimensional simplex $Fund_1$ with vertices $v_0, \dots, v_n \in \mathbb{R}^n$, where

$$v_i := (\underbrace{0, \dots, 0}_{i}, \underbrace{1, \dots, 1}_{n-i}) \quad (0 \leqslant i \leqslant n).$$

v_i is the intersection point of the reflecting hyperplanes for all generators of $\widetilde{C}_n$ except s_i.

The generators of St_n are $s_1, \dots, s_{n-1}$ acting as above, as well as

$$g_0(x_1, x_2, \dots, x_{n-1}, x_n) := (x_n - 2, x_2, \dots, x_{n-1}, x_1 + 2)$$

and

$$g_n = (s_n \cdots s_0)^{n+4}, \qquad (s_n \cdots s_0)(x_1, x_2, \dots, x_n) := (x_2, \dots, x_n, x_1 + 2).$$

Thus $g_0, s_1, \dots, s_{n-1}$ are reflections, while g_n is a cyclic permutation of coordinates combined with a translation. The reflecting hyperplanes for $g_0, s_1, \dots, s_{n-1}$ are

$$H'_0 := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = x_1 + 2\},$$
$$H_i := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_{i+1}\} \quad (1 \leqslant i \leqslant n-1).$$

Observation 3.5. The subgroup $\langle g_0, s_1, \dots, s_{n-1}\rangle$ of $\widetilde{C}_n$ is an affine Weyl group of type $\widetilde{A}_{n-1}$, and has

$$H := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$$

as an invariant subspace. A fundamental region for its action on H is the $(n-1)$-dimensional simplex Δ_{n-1} with vertices $w_0, \dots, w_{n-1} \in H$, where

$$w_i := \frac{2}{n}(\underbrace{i-n, \dots, i-n}_{i}, \underbrace{i, \dots, i}_{n-i}) \quad (0 \leqslant i \leqslant n-1).$$

Now note that

$$(s_n \cdots s_0)(w_i) = e + w_{i-1} \quad (0 \leqslant i \leqslant n-1)$$

with the index $i-1$ interpreted modulo n, and where

$$e = \frac{2}{n}(1, \dots, 1) \in H^{\perp}.$$

Therefore g_n acts as a linear transformation of H preserving Δ_{n-1}, combined with a translation by the vector $(n+4)e \in H^{\perp}$.

Observation 3.6. A fundamental region for the action of St_n on $\mathbb{R}^n$ is the prism

$$Fund_2 = \Delta_{n-1} \times I := \{ w + t(1,\dots,1) \mid w \in \Delta_{n-1},\ 0 \leqslant t \leqslant 2(n+4)/n \}.$$

Return now to the action of $\widetilde{C}_n$ on $CTFT(n)$. It can be verified that each of the generators of St_n stabilizes the canonical colored star triangulation T_0 defined immediately before Theorem 3.3, so that St_n is contained in the stabilizer of T_0 under the $\widetilde{C}_n$-action on $CTFT(n)$. In order to show that St_n is actually equal to this stabilizer, it suffices to show that both subgroups have the same finite index in $\widetilde{C}_n$. The index of the stabilizer is the size of the orbit of T_0, namely (by Proposition 3.2) the number of colored triangulations, $\#CTFT(n)$. The index of St_n in $\widetilde{C}_n$ is the quotient of volumes $vol(Fund_2)/vol(Fund_1)$. By Corollary 2.10 it thus suffices to show that

$$vol(Fund_2)/vol(Fund_1) = (n+4) \cdot 2^n.$$

This indeed follows from the following computations, using the well-known formula for the volume of a k-dimensional simplex Δ with vertices $v_0,\dots,v_k \in \mathbb{R}^k$:

$$vol(\Delta) = \frac{1}{k!} \cdot \left[\det\left(\langle v_i - v_0, v_j - v_0\rangle\right)_{1 \leqslant i,j \leqslant k}\right]^{1/2},$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^k$.

Claim 3.7.

$$vol(Fund_1) = \frac{1}{n!} \cdot \det(A)^{1/2},$$

where, following Observation 3.4*,*

$$A := (a_{ij}) \in \mathbb{R}^{n\times n},$$

$$a_{ij} := \langle v_i - v_0, v_j - v_0\rangle = \min(i,j) \quad (1 \leqslant i,j \leqslant n).$$

Claim 3.8.

$$vol(Fund_2) = \left[2(n+4)/n\right] \cdot n^{1/2} \cdot \frac{1}{(n-1)!} \cdot \det(B)^{1/2},$$

where, following Observations 3.5 *and* 3.6*,*

$$B := (b_{ij}) \in \mathbb{R}^{(n-1)\times(n-1)},$$

$$b_{ij} := \langle w_i - w_0, w_j - w_0\rangle = \frac{4}{n} \cdot \min(i,j) \cdot \min(n-i, n-j) \quad (1 \leqslant i,j \leqslant n-1).$$

Proof. For $i \leqslant j$,

$$\begin{aligned} b_{ij} &= \langle w_i - w_0, w_j - w_0 \rangle \\ &= \frac{4}{n^2} \cdot \big[i \cdot (i-n) \cdot (j-n) + (j-i) \cdot i \cdot (j-n) + (n-j) \cdot i \cdot j \big] \\ &= \frac{4}{n^2} \cdot \big[i \cdot (j-n)^2 + (n-j) \cdot i \cdot j \big] \\ &= \frac{4}{n^2} \cdot i \cdot (n-j) \cdot n. \qquad \square \end{aligned}$$

Claim 3.9.

$$\det(A) = 1$$

and

$$\det(B) = \left(\frac{4}{n}\right)^{n-1} \cdot n^{n-2} = 4^{n-1} \cdot n^{-1}.$$

Proof. By elementary row operations (subtracting row $i-1$ from row i, for $2 \leqslant i \leqslant n$), the $n \times n$ matrix $A = (\min(i, j))$ can be transformed into an upper triangular matrix with 1s in and over the main diagonal, so that $\det(A) = 1$.

By similar operations, the $(n-1) \times (n-1)$ matrix $(n/4) \cdot B = (\min(i, j) \cdot \min(n-i, n-j))$ can be transformed into the matrix $C = (c_{ij})$ with

$$c_{i,j} = \begin{cases} 1 \cdot (n-j), & \text{if } i \leqslant j; \\ j \cdot (-1), & \text{if } i > j. \end{cases}$$

Subtracting row $n-1$ from all the other rows we get the matrix $D = (d_{ij})$ with

$$d_{i,j} = \begin{cases} n, & \text{if } 1 \leqslant i \leqslant j \leqslant n-2; \\ 0, & \text{if } 1 \leqslant j < i \leqslant n-2; \\ 0, & \text{if } 1 \leqslant i \leqslant n-2 \text{ and } j = n-1; \\ -j, & \text{if } i = n-1 \text{ and } 1 \leqslant j \leqslant n-2; \\ 1, & \text{if } i = j = n-1. \end{cases}$$

It follows that $\det(C) = \det(D) = n^{n-2}$ and $\det(B) = 4^{n-1} \cdot n^{-1}$. $\square$

Claim 3.10.

$$vol(Fund_2)/vol(Fund_1) = (n+4) \cdot 2^n = \#CTFT(n).$$

Proof. By Claims 3.7, 3.8 and 3.9,

$$vol(Fund_1) = \frac{1}{n!}$$

while

$$vol(Fund_2) = 2(n+4)n^{-1/2} \cdot \frac{1}{(n-1)!} \cdot 2^{n-1} n^{-1/2} = \frac{1}{n!} \cdot 2^n (n+4). \qquad \square$$

This completes the proof of Theorem 3.3. $\square$

3.3. Coset representatives

The stabilizer St_n of the canonical colored star triangulation T_0 is not a parabolic subgroup of $\widetilde{C}_n$. However, it will be shown that a distinguished set of representatives of St_n in $\widetilde{C}_n$ forms an interval in the weak order on $\widetilde{C}_n$.

For $0 \leqslant i \leqslant n$ denote $a_i := s_i s_{i-1} \cdots s_0 \in \widetilde{C}_n$ and $b_i := s_{n-i} s_{n-i+1} \cdots s_n \in \widetilde{C}_n$.

Proposition 3.11. *Each of the sets*

$$R_n := \left\{a_0^{\epsilon_0} a_1^{\epsilon_1} \cdots a_{n-1}^{\epsilon_{n-1}} a_n^{\epsilon_n} \colon \epsilon_i \in \{0,1\}\ (0 \leqslant i < n) \text{ and } 0 \leqslant \epsilon_n < n+4\right\},$$
$$R'_n := \left\{b_0^{\epsilon_0} b_1^{\epsilon_1} \cdots b_{n-1}^{\epsilon_{n-1}} b_n^{\epsilon_n} \colon \epsilon_i \in \{0,1\}\ (0 \leqslant i < n) \text{ and } 0 \leqslant \epsilon_n < n+4\right\}$$

forms a complete list of representatives of the left cosets of St_n in $\widetilde{C}_n$.

Proof. Since $\#R_n \leqslant (n+4) \cdot 2^n$, in order to prove that R_n forms a complete list of coset representatives it suffices to prove that for every $T \in CTFT(n)$ there exists an element $r \in R_n$ such that $rT_0 = T$, where T_0 is the canonical colored star triangulation. By Observation 2.9, it suffices to prove that for every vector $\mathrm{v} = (v_0, \ldots, v_n) \in \mathbb{Z}_{n+4} \times \mathbb{Z}_2^n$ there exists $r \in R_n$ such that $\varphi(rT_0) = \mathrm{v}$. Indeed, by Observation 3.1,

$$\varphi\left(a_0^{v_n} a_1^{v_{n-1}} \cdots a_{n-1}^{v_1} a_n^{-v_0 \bmod (n+4)} T_0\right) = (v_0, \ldots, v_n).$$

The proof for R'_n is similar. $\square$

Let $\ell(w)$ be the length of an element $w \in \widetilde{C}_n$ with respect to the Coxeter generating set S (see Section 3.1 above), that is,

$$\ell(w) := \min\left\{\ell \colon w = s_{i_1} s_{i_2} \cdots s_{i_\ell},\ s_{i_j} \in \{s_0, \ldots, s_n\}\ (\forall j)\right\}.$$

Claim 3.12. *For every $r = a_0^{\epsilon_0} \cdots a_n^{\epsilon_n} \in R_n$*

$$\ell(r) = \sum_{j=0}^{n} (j+1)\epsilon_j = \sum_{j=0}^{n} \sum_{i=j}^{n} \epsilon_i.$$

Proof. Notice that for every $0 \leqslant i < n$, $a_{i+1}^{\epsilon_{i+1}}$ is a representative of shortest length of a right coset of the parabolic subgroup $\langle s_0, \ldots, s_i \rangle$ in $\langle s_0, \ldots, s_i, s_{i+1} \rangle$. The claim follows, by induction, from the length-additivity property of parabolic subgroups in Coxeter groups [5, §1.10], [1, §2.4]. $\square$

The following lemma plays a key role in understanding the structure of R_n (Proposition 3.15) and of the colored flip-graph (Proposition 4.1).

Lemma 3.13. *For every $r = a_0^{\epsilon_0} \cdots a_n^{\epsilon_n} \in R_n$ and a Coxeter generator s_i of $\widetilde{C}_n$ exactly one of the following holds:*

1. $s_i r \in R_n$.
2. $s_i r \in r\,St_n$.
3. (i) $i = n$, $\epsilon_{n-1} = 1$ *and* $\epsilon_n = n+3$. *Then* $s_n r \in a_0^{\epsilon_0} \cdots a_{n-2}^{\epsilon_{n-2}}\, St_n$.
   (ii) $i = n$, $\epsilon_{n-1} = 0$ *and* $\epsilon_n = 0$. *Then* $s_n r \in a_0^{\epsilon_0} \cdots a_{n-2}^{\epsilon_{n-2}} a_{n-1} a_n^{n+3}\, St_n$.

Corollary 3.14. *For every $s_i \in S$ and $r = a_0^{\epsilon_0} \cdots a_n^{\epsilon_n} \in R_n$*

$$\ell(s_i r) < \ell(r) \quad \Leftrightarrow \quad \epsilon_{i-1} = 0 \text{ and } \epsilon_i > 0,$$

where $\epsilon_{-1} := 0$.

For proofs of Lemma 3.13 and Corollary 3.14 see Appendix A.

Denote

$$w_0 := a_0 a_1 \cdots a_{n-1} a_n^{n+3}$$

the longest element in R_n.

Proposition 3.15. *R_n is a self-dual lower interval $\{w \in \widetilde{C}_n:\ id \leqslant w \leqslant w_0\}$ in the left weak order on $\widetilde{C}_n$; hence it forms a graded lattice.*

Proof. By Corollary 3.14, for every $r \in R_n$ and $s_i \in S$, $\ell(s_i r) < \ell(r)$ implies that $r = \cdots a_{i-1}^0 a_i^{\epsilon_i} \cdots$ ($\epsilon_i > 0$) for some $0 \leqslant i \leqslant n$, thus

$$s_i r = \cdots a_{i-1} a_i^{\epsilon_i - 1} \cdots \in R_n.$$

It follows that R_n is an interval in the left weak order.

Self-duality follows from the identity

$$r w_0 = a_0^{1-\epsilon_0} a_1^{1-\epsilon_1} \cdots a_{n-1}^{1-\epsilon_{n-1}} a_n^{n+3-\epsilon_n}$$

for all $r = a_0^{\epsilon_0} \cdots a_n^{\epsilon_n} \in R_n$. $\square$

Remark 3.16. Since R_n is an interval in the left weak order the rank of an element is given by its Coxeter length. Thus the rank generating function is

$$(1+q)\big(1+q^2\big)\cdots\big(1+q^n\big)\big(1+q^{n+1}+q^{2(n+1)}+\cdots+q^{(n+3)(n+1)}\big)$$

(not necessarily unimodal).

Lemma 3.17. *For every pair of elements in R_n*

$$a_0^{\epsilon_0} \cdots a_n^{\epsilon_n} < a_0^{\delta_0} \cdots a_n^{\delta_n}$$

in the left weak order if and only if

$$(\epsilon_n, \dots, \epsilon_0) < (\delta_n, \dots, \delta_0)$$

in the dominance order; i.e., $\sum_{i=k}^n \epsilon_i < \sum_{i=k}^n \delta_i$ for all $0 \leqslant k \leqslant n$.

Proof. By Corollary 3.14, the lemma holds for the covering relation. Proceed by induction on the length of the chain. $\square$

For every pair of elements $r, s \in R_n$ denote by $r \wedge s$ their join and by $r \vee s$ their meet in the weak order on $\widetilde{C}_n$. Lemma 3.17 implies

Corollary 3.18. *For every pair of elements in* R_n

$$a_0^{\epsilon_0}\cdots a_n^{\epsilon_n}\wedge a_0^{\delta_0}\cdots a_n^{\delta_n}=a_0^{\alpha_0}\cdots a_n^{\alpha_n},$$

where

$$\alpha_k:=\min\left\{\sum_{i=k}^n\epsilon_i,\sum_{i=k}^n\delta_i\right\}-\min\left\{\sum_{i=k+1}^n\epsilon_i,\sum_{i=k+1}^n\delta_i\right\}\quad(0\leqslant k\leqslant n),$$

and

$$a_0^{\epsilon_0}\cdots a_n^{\epsilon_n}\vee a_0^{\delta_0}\cdots a_n^{\delta_n}=a_0^{\beta_0}\cdots a_n^{\beta_n},$$

where

$$\beta_k:=\max\left\{\sum_{i=k}^n\epsilon_i,\sum_{i=k}^n\delta_i\right\}-\max\left\{\sum_{i=k+1}^n\epsilon_i,\sum_{i=k+1}^n\delta_i\right\}\quad(0\leqslant k\leqslant n).$$

It follows that

Corollary 3.19. *R_n forms a modular lattice with respect to the weak order; namely, for every $r,s\in R_n$*

$$\ell(r\vee s)+\ell(r\wedge s)=\ell(r)+\ell(s).$$

It should be noted that the weak order on $\widetilde{C}_n$ is not modular.

Proof. Combining Corollary 3.18 with Claim 3.12 yields

$$\ell(r\vee s)=\sum_{j=0}^n\sum_{i=j}^n\beta_i=\sum_{j=0}^n\max\left\{\sum_{i=j}^n\epsilon_i,\sum_{i=j}^n\delta_i\right\},$$

and, similarly,

$$\ell(r\wedge s)=\sum_{j=0}^n\min\left\{\sum_{i=j}^n\epsilon_i,\sum_{i=j}^n\delta_i\right\}.$$

Hence

$$\begin{aligned}\ell(r\wedge s)+\ell(r\vee s)&=\sum_{j=0}^n\max\left\{\sum_{i=j}^n\epsilon_i,\sum_{i=j}^n\delta_i\right\}+\sum_{j=0}^n\min\left\{\sum_{i=j}^n\epsilon_i,\sum_{i=j}^n\delta_i\right\}\\&=\sum_{j=0}^n\sum_{i=j}^n(\epsilon_i+\delta_i)=\ell(r)+\ell(s).\qquad\square\end{aligned}$$

4. The flip graph: algebraic description

The colored flip graph Γ_n is isomorphic to the Schreier graph of the cosets of St_n in $\widetilde{C}_n$ with respect to the Coxeter generating set $\{s_0, \dots, s_n\}$. Furthermore, fixing a set of coset representatives we can get an explicit description of Γ_n.

Proposition 4.1. *The colored flip graph Γ_n is isomorphic to the graph whose vertices are the elements in R_n; two distinct elements $r_1, r_2 \in R_n$ forms an edge if their quotient is a Coxeter generator of $\widetilde{C}_n$ or they are of the form $(v, va_{n-1}a_n^{n+3})$, for any $v = a_0^{\epsilon_0} \cdots a_{n-2}^{\epsilon_{n-2}}$.*

In other words, the flip graph is obtained from the (undirected) Hasse diagram Σ_n of the left weak order on R_n by adding the edges $(v, va_{n-1}a_n^{n+3})$, for any $v = a_0^{\epsilon_0} \cdots a_{n-2}^{\epsilon_{n-2}}$.

Proof. Proposition 4.1 is an immediate consequence of Lemma 3.13. $\square$

Observation 4.2. A right multiplication by a_n is an automorphism of the colored flip graph Γ_n.

Proof. A rotation by $\frac{2\pi}{n+4}$ of the colored triangulation $a_0^{\epsilon_0} \cdots a_{n-1}^{\epsilon_{n-1}} a_n^t T_0$ gives the triangulation $a_0^{\epsilon_0} \cdots a_{n-1}^{\epsilon_{n-1}} a_n^{t+1 \pmod{n+4}} T_0$. $\square$

For every pair $\pi, \sigma \in R_n$ let $\mathrm{dist}_{\Gamma_n}(\pi, \sigma)$ be the distance between πT_0 and σT_0 in Γ_n.
It follows from Observation 4.2 that

Corollary 4.3. *For every pair $r, s \in R_n$ and an integer t*

$$\mathrm{dist}_{\Gamma_n}(r, s) = \mathrm{dist}_{\Gamma_n}\left(ra_n^t, sa_n^t\right).$$

5. The flip graph: diameter

Denote by $\mathrm{Diam}(\Gamma_n)$ the diameter of the colored flip graph Γ_n.

Theorem 5.1. *For every $n > 0$*

$$\mathrm{Diam}(\Gamma_n) = \frac{(n+1)(n+4)}{2}.$$

5.1. Proof of Theorem 5.1

The proof relies on the intimate relation between the colored flip graph Γ_n and the Hasse diagram of the weak order on R_n, see Proposition 4.1 and comment afterwards. The upper bound (Lemma 5.4) is obtained by combining the properties of the weak order on R_n with the invariance of the flip graph under rotation. The grading of the Hasse diagram together with Proposition 4.1 implies a lower bound (Lemma 5.5).

5.1.1. Distance

For a graph G denote by $\mathrm{dist}_G(v, u)$ the distance (i.e., the length of the shortest path) between the vertices u and v. We begin with a general lemma.

Lemma 5.2. *Let P be a modular lattice. Let ℓ be its rank function and Σ its Hasse diagram. Then for every pair $r, s \in P$*

$$\mathrm{dist}_\Sigma(r, s) = \ell(r \vee s) - \ell(r \wedge s).$$

Proof. If there is a shortest path between r and s with at most one peak (local maximum), then by the modularity

$$\mathrm{dist}_\Sigma(r,s) = 2\ell(r \vee s) - \ell(r) - \ell(s) = \ell(r \vee s) - \ell(r \wedge s).$$

Given a shortest path from r to s with $k > 1$ peaks let v, w be two consequent peaks in the path. There is a unique local minimum z in the path from v to w. By the minimality of the length of the path, $z = v \wedge w$. By the modularity we can replace the segment from v to w through the meet z by a path through $v \wedge w$ and obtain a path of same length and $k-1$ peaks. Proceed by recursion to get a shortest path with one peak. $\square$

Lemma 5.3. *For every pair* $r = \prod_{i=0}^n a_i^{\epsilon_i}$, $s = \prod_{i=0}^n a_i^{\delta_i} \in V(\Gamma_n) = R_n$

$$\mathrm{dist}_{\Gamma_n}(r,s) = \min\bigl\{\ell\bigl(ra_n^{-\epsilon_n} \vee sa_n^{-\epsilon_n}\bigr) - \ell\bigl(ra_n^{-\epsilon_n} \wedge sa_n^{-\epsilon_n}\bigr), \ell\bigl(ra_n^{-\delta_n} \vee sa_n^{-\delta_n}\bigr) - \ell\bigl(ra_n^{-\delta_n} \wedge sa_n^{-\delta_n}\bigr)\bigr\}.$$

Proof. Let C_n be a cycle of length $n+4$ whose set of vertices is $\{u_i\colon\ 0 \leqslant i < n+4\}$ and edges $(u_i, u_{(i+1) \bmod (n+4)})$ for every $0 \leqslant i < n+4$. Consider the map $\rho : \Gamma_n \to C_n$, defined by $\rho(a_0^{\epsilon_0} \cdots a_{n-1}^{\epsilon_{n-1}} a_n^i) := u_i$. By Proposition 4.1, ρ is a graph homomorphism. Let U_i be the pre-image of u_i, i.e.

$$U_i := \rho^{-1}(u_i) = \bigl\{a_0^{\epsilon_0} \cdots a_{n-1}^{\epsilon_{n-1}} a_n^i\colon\ \epsilon_j \in \{0,1\} \text{ for all } 0 \leqslant j < n\bigr\} \quad (0 \leqslant i < n+4).$$

Notice that the subgraph of Γ_n induced by U_i is isomorphic to the undirected Hasse diagram of the weak order on U_i.

We first claim that for any $r, s \in R_n$ a shortest path from r to s in Γ_n does not contain a sequence of the form $v_1, \ldots, v_k$, where $v_1 = \prod_{j=0}^{n-1} a_j^{\mu_j} a_n^i$, $v_k = \prod_{j=0}^{n-1} a_j^{\nu_j} a_n^i \in U_i$ for some i and $v_2, \ldots, v_{k-1} \in U_j$ for $j = (i \pm 1) \bmod (n+4)$. If there is such a shortest path, then by Corollary 4.3, we may assume that $i = 0$ and $j = 1$; namely $v_1, v_k \in U_0$ and $v_1, v_2, \ldots, v_{k-1} \in U_1$. By assumption of the length minimality of the path

$$\mathrm{dist}_{U_0}(v_1, v_k) \geqslant 2 + \mathrm{dist}_{U_1}(v_2, v_{k-1}).$$

On the other hand, by Proposition 4.1, $\mu_{n-1} = \nu_{n-1} = 1$, $v_2 = \prod_{j=0}^{n-2} a_j^{\mu_j} a_n$, and $v_{k-1} = \prod_{j=0}^{n-2} a_j^{\nu_j} a_n$. Hence, by Lemma 5.2 together with Corollary 3.18,

$$\mathrm{dist}_{U_0}(v_1, v_k) = \mathrm{dist}_{U_1}(v_2, v_{k-1}).$$

Contradiction.

We deduce that the ρ-image of the shortest path between any pair $r, s \in U_i$ for some i is either of length zero or a multiple of a full cycle. But it cannot be a multiple of a full cycle since a pre-image of a full cycle is of length at least

$$\ell\bigl(a_n^{n+3}\bigr) - \ell(a_0 \cdots a_{n-1}) = \frac{(n+1)(n+5)}{2}.$$

On the other hand, by Lemma 5.2, since U_i is a modular lattice, the diameter of U_i is the difference between the lengths of the top and bottom elements in U_i. That is

$$\mathrm{Diam}(U_i) = \ell(a_0 \cdots a_{n-1} a_i) - \ell(a_i) = \binom{n+1}{2}. \tag{5}$$

One concludes that for any pair $r, s \in U_i$ the shortest path is contained in the modular lattice U_i, so the lemma holds for such a pair.

If $r \in U_i$, $s \in U_j$ and $i < j$ then by the above arguments the ρ-image of the shortest path between r and s is one of the two intervals from u_i to u_j in the cycle. By Corollary 4.3, a right multiplication (by $a_n^{-\epsilon_n}$ if the image contains u_{i+1} or by $a^{-\delta_n}$ otherwise) maps the shortest path to a shortest path in the modular lattice R_n. Lemma 5.2 completes the proof. $\square$

5.1.2. Diameter: upper bound

In this subsection we prove

Lemma 5.4. *For every $n > 0$*

$$\mathrm{Diam}(\Gamma_n) \leqslant \frac{(n+1)(n+4)}{2}.$$

Proof. For $n = 1, 2$ the proof is direct: Γ_1 is a cycle of length 10, thus its diameter is 5; for $n = 2$ see Fig. 2.1.

It suffices to prove that the lemma holds for $n > 2$. By the lattice property and modularity of R_n (Corollary 3.19) together with Claim 3.12 and Corollary 3.18, for every $n > 2$ and $r = a_0^{\epsilon_0} \cdots a_n^{\epsilon_n}$, $s = a_0^{\delta_0} \cdots a_n^{\delta_n} \in R_n$

$$\mathrm{dist}_{\Gamma_n}(r, s) = \ell(r \vee s) - \ell(r \wedge s) = \sum_{j=0}^{n} \left| \sum_{i=j}^{n} (\epsilon_i - \delta_i) \right|. \tag{6}$$

If $\epsilon_n = \delta_n$ then there exists $0 \leqslant i < n+4$ such that $r, s \in U_i$. Then by (5),

$$\mathrm{dist}_{\Gamma_n}(r, s) \leqslant \binom{n+1}{2}.$$

If $\epsilon_n \neq \delta_n$ then by Corollary 4.3, we may assume, without loss of generality, that $\delta_n = 0$. Also, by note that Corollary 4.3, $\mathrm{dist}_{\Gamma_n}(r, s) = \mathrm{dist}(ra_n^{-\epsilon_n}, sa_n^{-\epsilon_n})$. Now, by Lemma 5.3 together with (6) and the assumption $\delta_n = 0$,

$$\mathrm{dist}_{\Gamma_n}(r, s) = \min\left\{ \sum_{j=0}^{n} \left| \epsilon_n + \sum_{i=j}^{n-1} (\epsilon_i - \delta_i) \right|, \sum_{j=0}^{n} \left| n+4-\epsilon_n - \sum_{i=j}^{n-1} (\epsilon_i - \delta_i) \right| \right\}. \tag{7}$$

For $0 \leqslant j \leqslant n$ denote $x_j := \epsilon_n + \sum_{i=j}^{n} (\epsilon_i - \delta_i) - \frac{n+4}{2}$. Then

$$\mathrm{dist}_{\Gamma_n}(r, s) = \min\left\{ \sum_{j=0}^{n} \left| \frac{n+4}{2} + x_j \right|, \sum_{j=0}^{n} \left| \frac{n+4}{2} - x_j \right| \right\},$$

where, by definition, (i) $-\frac{n+4}{2} \leqslant x_n < \frac{n+4}{2}$ and (ii) $|x_{j+1} - x_j| \leqslant 1$.

By (i), $\frac{n+4}{2} + x_n \geqslant 0$. Combining this with (ii) implies that if $\frac{n+4}{2} + x_j$ is negative for some j, then there exists $0 \leqslant j_0 \leqslant n$, such that $\frac{n+4}{2} + x_{j_0} = 0$. Then, by (ii), for every $0 \leqslant j \leqslant n$, $|\frac{n+4}{2} + x_j| \leqslant |j - j_0|$. Hence $\sum_{j=0}^{n} |\frac{n+4}{2} + x_j| \leqslant \binom{n+1}{2}$. So, we may assume that $\frac{n+4}{2} + x_j$ is positive for all $0 \leqslant j \leqslant n$. By a similar reasoning (regarding the second sum), we may assume that $\frac{n+4}{2} - x_j$ is positive for all $0 \leqslant j \leqslant n$. Thus

$$\mathrm{dist}_{\Gamma_n}(r,s)=\min\left\{\sum_{j=0}^{n}\frac{n+4}{2}+x_j,\ \sum_{j=0}^{n}\frac{n+4}{2}-x_j\right\}\leqslant\frac{(n+1)(n+4)}{2}. \qquad \square$$

5.1.3. Diameter: lower bound

In this subsection we prove

Lemma 5.5. *For every $n>0$*

$$\mathrm{Diam}(\Gamma_n)\geqslant\frac{(n+1)(n+4)}{2}.$$

Proof. Again, for $n=1,2$ the proof is direct (for $=2$ see Fig. 2.1).

It suffices to prove that for every $n>2$ and $r\in R_n$ there exists an element $s\in R_n$ such that

$$\mathrm{dist}_{\Gamma_n}(r,s)\geqslant\frac{(n+1)(n+4)}{2}.$$

Since the Hasse diagram Σ_n on R_n is graded by the length function ℓ, and since Γ_n is obtained from the Σ_n by adding the edges $(v, va_{n-1}a_n^{n+3})$, for any $v=a_0^{\epsilon_0}\cdots a_{n-2}^{\epsilon_{n-2}}$ (Proposition 4.1) it follows that for every $r,s\in R_n$

$$\mathrm{dist}_{\Gamma_n}(r,s)\geqslant\min\bigl\{\bigl|\ell(s)-\ell(r)\bigr|,\ \ell\bigl(a_{n-1}a_n^{n+3}\bigr)+1-\bigl|\ell(s)-\ell(r)\bigr|\bigr\}. \tag{8}$$

It follows that

$$\begin{aligned}\mathrm{Diam}(\Gamma_n)&=\max\bigl\{\mathrm{dist}_{\Gamma_n}(r,s)\colon\ r,s\in R_n\bigr\}\\&\geqslant\max\left\{\min\bigl\{d,\ (n+1)(n+4)-d\bigr\}\colon\ 0\leqslant d\leqslant 3\binom{n+2}{2}\right\},\end{aligned}$$

where $d:=|\ell(s)-\ell(r)|$, hence $0\leqslant d\leqslant\ell(w_0)=\binom{n}{2}+(n+3)(n+1)=3\binom{n+2}{2}$. By Proposition 3.15, for any given $r\in R_n$, there exists an $s\in R_n$ of length distance $\frac{(n+1)(n+4)}{2}$, completing the proof. $\square$

Combining Lemma 5.4 with Lemma 5.5 completes the proof of Theorem 5.1.

5.2. Antipodes

Let $\phi:\mathit{CTFT}(n)\to\mathit{CTFT}(n)$ denote the map which reverse the coloring of a triangle free triangulation; namely each color i is replaced by $n-i$. Clearly, ϕ is an automorphism of the colored flip graph Γ_n. Furthermore,

Proposition 5.6. *For every $T\in\mathit{CTFT}(n)$ the flip distance between T and $\phi(T)$ is equal to* $\mathrm{Diam}(\Gamma_n)$.

To prove that we need the following lemma. Let f be the natural bijection from $\mathit{CTFT}(n)$ to R_n: $f(T):=r$ if $rT_0=T$. Then

Lemma 5.7. *For every $r=a_0^{\epsilon_0}\cdots a_{n-1}^{\epsilon_{n-1}}a_n^{\epsilon_n}$ with $\epsilon_i\in\{0,1\}$ $(0\leqslant i<n)$ and $0\leqslant\epsilon_n<n+4$*

$$f^{-1}\phi(rT_0)=\prod_{i=0}^{n-1}a_i^{1-\epsilon_{n-1-i}}\cdot a_n^m,$$

where $m:=(2+\sum_{i=0}^{n}\epsilon_i)\pmod{n+4}$.

Proof of Proposition 5.6. By Corollary 4.3, we may assume that $\epsilon_n = 0$. Then by Lemma 5.7, $f^{-1}\phi(rT_0) = a_0^{1-\epsilon_{n-1}} \cdots a_{n-1}^{1-\epsilon_0} a_n^m$, where $m = 2 + \sum_{i=0}^{n-1} \epsilon_i$. By Claim 3.12,

$$\begin{aligned}\ell\big(f^{-1}\phi(rT_0)\big) - \ell(r) &= \left(2 + \sum_{i=0}^{n-1} \epsilon_i\right)(n+1) + \sum_{i=0}^{n-1}(i+1)(1-\epsilon_{n-1-i}) - \sum_{i=0}^{n-1}(i+1)\epsilon_i \\ &= 2(n+1) + \sum_{i=0}^{n-1}(i+1) + \sum_{i=0}^{n-1}\epsilon_i\big((n+1)-(i+1)-(n-i)\big) \\ &= \frac{(n+1)(n+4)}{2}.\end{aligned}$$

Hence by (8), $\mathrm{dist}_{\Gamma_n}(f^{-1}\phi(rT_0), r) \geqslant \frac{(n+1)(n+4)}{2}$, so it is equal to the diameter. $\square$

Another antipode may be obtained by rotation. For an even n let ψ denote the rotation of a triangle free triangulation $T \in CTFT(n)$ by π with respect to the center of P_{n+4}. Then

Proposition 5.8. *For every even n and $T \in CTFT(n)$ the flip distance between T and $\psi(T)$ is equal to* $\mathrm{Diam}(\Gamma_n)$.

Proof. Without loss of generality, $r = f^{-1}(T) = a_0^{\epsilon_0} \cdots a_{n-1}^{\epsilon_{n-1}}$. By the proof of Observation 4.2, for every $r \in R_n$ $f^{-1}\psi(rT_0) = ra_n^{(n+4)/2}$. Hence, by Claim 3.12,

$$\ell\big(f^{-1}\psi(rT_0)\big) - \ell(r) = \frac{n+4}{2}(n+1).$$

Combining this with (8) yields

$$\mathrm{dist}_{\Gamma_n}\big(f^{-1}\psi(rT_0), r\big) \geqslant \frac{n+4}{2}(n+1) = \mathrm{Diam}(\Gamma_n). \qquad \square$$

6. Final remarks

The colored flip-graph is bipartite; the bipartition is fixed by the parity of the length of the corresponding elements in R_n.

Recall the natural bijection $f : CTFT(n) \to R_n$, defined by $f(T) := r$ if $rT_0 = T$.

Definition 6.1. For every $T \in CTFT(n)$ associate a sign

$$\mathrm{sign}(T) := (-1)^{\ell(f(T))}.$$

A triangulation $T \in CTFT(n)$ is even if $\mathrm{sign}(T) = 1$ and odd otherwise.

Proposition 6.2. *The graph Γ_n is bipartite*; *the flip operation changes the sign.*

Proof. By Proposition 4.1, the vertices of Γ_n may be identified with elements in R_n, where for every pair $r, s \in R_n$ (r, s) is an edge in the colored flip graph if and only if rs^{-1} is a Coxeter generator or equals to $(a_{n-1}a_n^{n+3})^{\pm 1}$. Notice that for every n the length $\ell(a_{n-1}a_n^{n+3}) = n + (n+1)(n+3)$ is odd. We conclude that if (r, s) is an edge then r and s differ by the parity of their Coxeter length. $\square$

Proposition 6.3. *The number of even triangulations is equal to the number of odd triangulations.*

Proof. By definition of the sign, it suffices to show that there exists an invertible map from R_n to itself, which changes the parity of the length. A left multiplication by s_0 is such a map. $\square$

Hereby we mention (without proofs) some properties of the stabilizer St_n.

Proposition 6.4. *The stabilizer St_n is isomorphic to the direct product $\widetilde{A}_n \otimes \mathbb{Z}$.*

Even though St_n is not a parabolic subgroup of $\widetilde{C}_n$ the following remarkable property holds.

Proposition 6.5. *Every coset of St_n in $\widetilde{C}_n$ has a unique shortest representative.*

The set of shortest representatives may be constructed from R_n by a slight modification. Let $B(n,r) := \{\pi \in \widetilde{C}_n\colon\ \ell(\pi) \leqslant r\}$ be the ball of radius r in $\widetilde{C}_n$.

Proposition 6.6. *The set*

$$\hat{R}_n := \left(R_n \cap B\left(n, \frac{(n+1)(n+4)}{2}\right)\right) \cup \left(R_n \setminus B\left(n, \frac{(n+1)(n+4)}{2}\right)\right) g_n^{-1}$$

forms a complete list of shortest representatives of the left cosets of St_n in $\widetilde{C}_n$.

Finally, the evaluation of the diameter of the flip graph of uncolored triangle-free triangulations involves a surprisingly subtle optimization problem. Let G_n be the graph whose nodes are all uncolored triangle-free triangulations of a convex $(n+4)$-gon, where two nodes are connected by an arc if one is obtained from the other by a flip of a chord. Recall the map ϕ from Section 5.2 above. By definition, G_n is isomorphic to Γ_n/ϕ. It follows that

$$\frac{\mathrm{Diam}(\Gamma_n)}{2} \leqslant \mathrm{Diam}(G_n) \leqslant \mathrm{Diam}(\Gamma_n).$$

Preliminary computations hint on the following conjecture.

Conjecture 6.7. *The diameter of G_n is asymptotically equals to $(1 - \frac{1}{\sqrt{2}})n^2$.*

Appendix A. Proofs of Lemma 3.13 and Corollary 3.14

Proof of Lemma 3.13. First, notice that, by the braid relations of $\widetilde{C}_n$ (letting $a_{-1} := id$)

$$s_i a_j = a_j s_i \quad (-2 \leqslant j-1 < i \leqslant n); \tag{9}$$

$$s_i a_i = a_{i-1} \quad \text{and} \quad s_i a_{i-1} = a_i \quad (0 \leqslant i \leqslant n); \tag{10}$$

$$s_i a_j = a_j s_{i+1} \quad (0 < i < j \text{ and } i \neq n-1); \tag{11}$$

and

$$s_{n-1} a_n = a_n g_0, \tag{12}$$

and

$$a_n s_1 = g_0 a_n, \tag{13}$$

where $g_0 := s_0 s_1 \cdots s_{n-2} s_n s_{n-1} s_n s_{n-2} \cdots s_1 s_0$. Recall that $g_0 \in St_n$ (see Theorem 3.3).

We proceed by cases analysis.

Case (a). $\epsilon_{i-1}=0$ and $\epsilon_i>0$ (where $\epsilon_{-1}:=0$).

By (10) and (9),

$$\begin{aligned} s_i r &= s_i a_0^{\epsilon_0}\cdots a_{i-2}^{\epsilon_{i-2}} a_i^{\epsilon_i} a_{i+1}^{\epsilon_{i+1}}\cdots a_n^{\epsilon_n} = a_0^{\epsilon_0}\cdots a_{i-2}^{\epsilon_{i-2}} s_i a_i a_i^{\epsilon_i-1} a_{i+1}^{\epsilon_{i+1}}\cdots a_n^{\epsilon_n} \\ &= a_0^{\epsilon_0}\cdots a_{i-2}^{\epsilon_{i-2}} a_{i-1} a_i^{\epsilon_i-1} a_{i+1}^{\epsilon_{i+1}}\cdots a_n^{\epsilon_n}\in R_n. \end{aligned}$$

Case (b). $0<i<n$, $\epsilon_{i-1}=1$ and $\epsilon_i=0$, or $i=0$ and $\epsilon_0=0$.

Since $s_i^2=1$, it follows from the analysis of the previous case that in this case

$$s_i r = a_0^{\epsilon_0}\cdots a_{i-2}^{\epsilon_{i-2}} a_i a_{i+1}^{\epsilon_{i+1}}\cdots a_n^{\epsilon_n}\in R_n.$$

Case (c). $i=n$.

If $\epsilon_{n-1}=1$ then $s_n r = a_0^{\epsilon_0}\cdots a_{n-2}^{\epsilon_{n-2}} a_n^{\epsilon_n+1}$. For $\epsilon_n<n+3$ this is an element in R_n. If $\epsilon_n=n+3$ then, since $a_n^{n+4}\in St_n$, $s_n r\in a_0^{\epsilon_0}\cdots a_{n-2}^{\epsilon_{n-2}}\, St_n$.

If $\epsilon_{n-1}=0$ then $s_n r = a_0^{\epsilon_0}\cdots a_{n-2}^{\epsilon_{n-2}} a_{n-1} a_n^{\epsilon_n-1}$. This element is in R_n if $\epsilon_n>0$, and belongs to $a_0^{\epsilon_0}\cdots a_{n-2}^{\epsilon_{n-2}} a_{n-1} a_n^{\epsilon_n+3}\, St_n$ otherwise.

Case (d). $0<i<n$, $\epsilon_{i-1}=1$ and $\epsilon_i=1$.

By the braid relations of $\widetilde{C}_n$, for every $0<i<n$, $s_i a_{i-1} a_i = a_i^2 = a_{i-1} a_i s_1$. Hence

$$\begin{aligned} s_i r &= s_i a_0^{\epsilon_0}\cdots a_{i-2}^{\epsilon_{i-2}} a_{i-1} a_i a_{i+1}^{\epsilon_{i+1}}\cdots a_n^{\epsilon_n} = a_0^{\epsilon_0}\cdots a_{i-2}^{\epsilon_{i-2}} s_i a_{i-1} a_i a_{i+1}^{\epsilon_{i+1}}\cdots a_n^{\epsilon_n} \\ &= a_0^{\epsilon_0}\cdots a_{i-2}^{\epsilon_{i-2}} a_{i-1} a_i s_1 a_{i+1}^{\epsilon_{i+1}}\cdots a_n^{\epsilon_n} = r s_1\in r\, St_n. \end{aligned}$$

Case (e). $0<i<n$, $\epsilon_{i-1}=0$ and $\epsilon_i=0$.

By (9)

$$s_i r = a_0^{\epsilon_0}\cdots a_{i-2}^{\epsilon_{i+2}} s_i a_{i+1}^{\epsilon_{i+1}}\cdots a_n^{\epsilon_n} = a_0^{\epsilon_0}\cdots a_{i-2}^{\epsilon_{i+2}} a_{i+1}^{\epsilon_{i+1}}\cdots a_{n-1}^{\epsilon_{n-1}} s_{i+k} a_n^{\epsilon_n},$$

where $k:=\#\{j:\ i<j<n,\ \epsilon_j=1\}$.

If $i+k<n-\epsilon_n$ then, by (11), $s_{i+k}a_n^{\epsilon_n}=a_n^{\epsilon_n}s_{i+k+\epsilon_n}$, so that $s_i r = r s_{i+k+\epsilon_n}$. Since $0<i+k+\epsilon_n<n$, $s_{i+k+\epsilon_n}\in St_n$, thus $s_i r\in r\, St_n$.

If $i+k\geqslant n-\epsilon_n$, then by definition of k, $i+k=n-1$. By (12), (13) and (11),

$$s_{i+k}a_n^{\epsilon_n} = a_n g_0 a_n^{\epsilon_n-1} = \begin{cases} a_n^{\epsilon_n} g_0, & \text{if } \epsilon_n=1,\\ a_n^{\epsilon_n} s_{\epsilon_n-1}, & \text{if } 1<\epsilon_n\leqslant n,\\ a_n^{\epsilon_n} g_0, & \text{if } \epsilon_n=n+1,\\ a_n^{\epsilon_n} s_{m-1}, & \text{if } \epsilon_n=n+m,\ m=2,3. \end{cases}$$

Hence $s_i r\in r\, St_n$. $\square$

Proof of Corollary 3.14. The proof follows from the case-by-case analysis in the proof of Lemma 3.13. If $\epsilon_{i-1} = 0$ and $\epsilon_i > 0$ then $s_i r = s_i(\cdots a_{i-1}^0 a_i^{\epsilon_i} \cdots) = \cdots a_{i-1} a_i^{\epsilon_i - 1} \cdots$. By Claim 3.12, $\ell(s_i r) < \ell(r)$. If $\epsilon_{i-1} = 1$ and $\epsilon_i = 0$, by same argument $\ell(s_i r) > \ell(r)$. Similarly, for $i = n$ and $\epsilon_{n-1} = 1$. Otherwise, by Lemma 3.13, $s_i r = rg$ for some $g \in St_n$. By the length-additivity property [5, §1.10], $\ell(s_i r) = \ell(r) + \ell(g) \geqslant \ell(r)$. $\square$

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